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About the spacing functions of the three matrix ensembles

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Abstract. Three ensembles of random matrices have been extensively studied, orthogonal, unitary and symplectic, characterized by a parameter β taking values 1, 2 and 4. The probability $E_\beta(r,t)$ that a randomly chosen interval of length $2t$ contains exactly r eigenvalues of such a matrix can be expressed in terms of the rth partial derivative of a Fredholm determinant. Using this fact we give a new proof of some known relations between $E_1(r,t)$, $E_2(r,t)$ and $E_4(r,t)$, as well as a relation between odd and even spheroidal functions.

1. Introduction

In the study of random matrices much attention has been given to three Gaussian ensembles or three circular ensembles, named orthogonal, unitary and symplectic [1]. They are characterized by a parameter β taking the values 1, 2 and 4, respectively. The probability $E_\beta(r,t)$ that a randomly chosen interval of length $2t$ contains exactly r eigenvalues of a random matrix taken from one of these ensembles can be expressed as [2]

$$E_\beta(r,t) = \frac{1}{r!}\left(-\frac{\partial}{\partial z}\right)^r F_\beta(z,t)|_{z=1} \tag{1.1}$$

$$F_\beta(z,t) = \prod_{j=0}^{\infty}(1 - z\lambda_j) \tag{1.2}$$

where $\lambda_j = \lambda_j(t)$ are the eigenvalues of an integral equation

$$\lambda f(x) = \int_{-t}^{t} K(\beta; x, y) f(y)\,\mathrm{d}y \tag{1.3}$$

with the kernels [2]

$$K(2; x, y) = S(x, y) \tag{1.4}$$

$$K(1; x, y) = \begin{bmatrix} S(x,y) & D(x,y) \\ J(x,y) & S(x,y) \end{bmatrix} \tag{1.5}$$

$$K(4; x, y) = \begin{bmatrix} S(2x,2y) & D(2x,2y) \\ I(2x,2y) & S(2x,2y) \end{bmatrix}. \tag{1.6}$$

For $\beta = 2$, the kernel is simple, while for $\beta = 1$ and $\beta = 4$ the kernels are 2×2 matrices and the integral equation (1.3) is a set of two coupled equations.

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For $\beta = 1$ and $\beta = 4$, the eigenvalues are doubly degenerate because the kernel is self-dual in the quaternion sense [1]

$$K^{\mathrm{T}}(\beta; y, x) = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} K(\beta; x, y) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{1.7}$$

In equation (1.2) the product is taken over distinct eigenvalues λ_j.

In the limit of very large matrices one has

$$S(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)} \tag{1.8}$$

$$D(x, y) = \frac{\partial}{\partial x} S(x, y) \tag{1.9}$$

$$I(x, y) = \int_{-\infty}^{\infty} \epsilon(x-\xi) S(\xi, y)\, \mathrm{d}\xi \tag{1.10}$$

$$J(x, y) = I(x, y) - \epsilon(x-y) \tag{1.11}$$

and

$$\epsilon(x) = \begin{cases} +1/2 & \text{for } x > 0 \\ -1/2 & \text{for } x < 0. \end{cases} \tag{1.12}$$

In the case $\beta = 2$, the solutions of equation (1.3) are either even or odd. They are also the solutions of

$$\lambda f(x) = \int_{-t}^{t} S_{\pm}(x, y) f(y)\, \mathrm{d}y \tag{1.13}$$

with

$$S_{\pm}(x, y) = \tfrac{1}{2}[S(x, y) \pm S(x, -y)]. \tag{1.14}$$

We will use an even (odd) index to λ when it corresponds to an even (odd) solution, and write equation (1.2) as

$$F_2(z, t) = F_+(z, t) F_-(z, t) \tag{1.15}$$

$$F_+(z, t) = \prod_{i=0}^{\infty} (1 - z\lambda_{2i}) \tag{1.16}$$

$$F_-(z, t) = \prod_{i=0}^{\infty} (1 - z\lambda_{2i+1}). \tag{1.17}$$

In the case $\beta = 1$ or $\beta = 4$, each component of any eigenfunction has a definite parity, and the parities of the two components of an eigenfunction are opposite.

If one sets

$$E_{\pm}(r, t) = \frac{1}{r!} \left(-\frac{\partial}{\partial z} \right)^r F_{\pm}(z, t)|_{z=1} \tag{1.18}$$

or

$$F_{\pm}(z, t) = \sum_{r=0}^{\infty} (1-z)^r E_{\pm}(r, t) \tag{1.19}$$

then one knows that [3]

$$E_{\pm}(r, t) = E_1(2r, t) + E_1(2r \mp 1, t) \qquad r \geqslant 0 \tag{1.20}$$

$$E_4(r, t) = \tfrac{1}{2}[E_+(r, 2t) + E_-(r, 2t)] \qquad r \geqslant 0 \tag{1.21}$$

($E_1(-1, t) \equiv 0$). We will present here a new proof of equations (1.20) and (1.21).

Equation (1.8) and onwards have been written for very large matrices. They are the limit when $n \to \infty$ of the results for $n \times n$ matrices either from the Gaussian ensembles or from the circular ensembles. For the case of finite $n \times n$ matrices from the circular ensembles, equation (1.8) is replaced by [1]

$$S(\theta, \phi) = S(\theta - \phi) = \frac{1}{n} \sum_p \mathrm{e}^{\mathrm{i}p(\theta-\phi)} = \frac{\sin n(\theta - \phi)/2}{n \sin(\theta - \phi)/2} \tag{1.22}$$

and in all other equations x, y and t are replaced by θ, ϕ and α, respectively. Instead of an infinite number of eigenvalues we have a finite number n of them. For $\beta = 2$, the number of even (odd) eigenfunctions is $[(n+1)/2]$ ($[n/2]$). The even (odd) eigenfunctions are also the eigenfunctions of $S_+(\theta, \phi)$ ($S_-(\theta, \phi)$). In equation (1.22) p is summed over the values $-(n-1)/2, -(n-3)/2, \ldots, (n-3)/2, (n-1)/2$. Note that $S(\theta, \phi)$ depends only on the difference $\theta - \phi$, as indicated by the second expression in equation (1.22) above. Similarly $S(x, y)$ depends only on $x - y$.

It is somewhat convenient to argue when n is finite and later take the limit $n \to \infty$ while $t = n\alpha/2\pi$, $x = n\theta/2\pi$, $y = n\phi/2\pi$ are kept finite.

2. Relation between odd and even solutions of equation (1.13)

The derivatives of the eigenfunctions of equation (1.3) for the three cases $\beta = 1$, $\beta = 2$ and $\beta = 4$ can be expanded in terms of the eigenfunctions themselves for the case $\beta = 2$. This can be seen directly from the integral equation for the finite n case, since the kernel is a sum of separable functions involving exponentials. Thus

$$f'_{2i+1}(\theta) = \sum_j c_{ij} f_{2j}(\theta) \qquad f_{2i+1}(\theta) = \sum_j c_{ij} \int_0^\theta f_{2j}(\phi)\, \mathrm{d}\phi \tag{2.1}$$

$$f'_{2i}(\theta) = \sum_j d_{ij} f_{2j+1}(\theta) \qquad f_{2i}(\theta) = f_{2i}(0) + \sum_j d_{ij} \int_0^\theta f_{2j+1}(\phi)\, \mathrm{d}\phi. \tag{2.2}$$

The kernels $S_\pm(\theta, \phi)$ satisfy the obvious property

$$\frac{\partial}{\partial\theta} S_\pm(\theta, \phi) = -\frac{\partial}{\partial\phi} S_\mp(\theta, \phi) \tag{2.3}$$

and have the spectral representations

$$S_+(\theta, \phi) = \sum_i \lambda_{2i} f_{2i}(\theta) f_{2i}(\phi) \tag{2.4}$$

$$S_-(\theta, \phi) = \sum_i \lambda_{2i+1} f_{2i+1}(\theta) f_{2i+1}(\phi) \tag{2.5}$$

where $f_j(\theta)$ are normalized eigenfunctions of equation (1.13)

$$\int_{-\alpha}^{\alpha} f_i(\theta) f_j(\theta)\, \mathrm{d}\theta = \delta_{ij}. \tag{2.6}$$

Differentiation of equation (1.13) and a partial integration gives

$$\begin{aligned} \lambda_{2i+1} f'_{2i+1}(\theta) &= \frac{\partial}{\partial\theta} \int_{-\alpha}^{\alpha} S_-(\theta, \phi) f_{2i+1}(\phi)\, \mathrm{d}\phi \\ &= -\int_{-\alpha}^{\alpha} \frac{\partial}{\partial\phi} S_+(\theta, \phi) f_{2i+1}(\phi)\, \mathrm{d}\phi \\ &= -2S_+(\theta, \alpha) f_{2i+1}(\alpha) + \int_{-\alpha}^{\alpha} S_+(\theta, \phi) f'_{2i+1}(\phi)\, \mathrm{d}\phi \end{aligned} \tag{2.7}$$

or using equations (2.1), (2.4) and (2.6),

$$\lambda_{2i+1} \sum_j c_{ij} f_{2j}(\theta) = -2 f_{2i+1}(\alpha) \sum_j \lambda_{2j} f_{2j}(\theta) f_{2j}(\alpha) + \sum_j c_{ij} \lambda_{2j} f_{2j}(\theta). \tag{2.8}$$

As $f_{2j}(\theta)$ are linearly independent even functions, one has

$$c_{ij} = \frac{-2\lambda_{2j}}{\lambda_{2i+1} - \lambda_{2j}} f_{2j}(\alpha) f_{2i+1}(\alpha) \tag{2.9}$$

and equation (2.1) can be written as

$$\begin{aligned} f_{2i+1}(\alpha) &= \sum_j c_{ij} \int_0^\alpha f_{2j}(\phi)\,\mathrm{d}\phi \\ &= f_{2i+1}(\alpha) \sum_j \frac{-2\lambda_{2j}}{\lambda_{2i+1} - \lambda_{2j}} f_{2j}(\alpha) \int_0^\alpha f_{2j}(\phi)\,\mathrm{d}\phi. \end{aligned} \tag{2.10}$$

Since $f_{2i+1}(\alpha) \neq 0$ from equation (2.9), one has

$$1 + \sum_j \frac{2\lambda_{2j}}{\lambda_{2i+1} - \lambda_{2j}} f_{2j}(\alpha) \int_0^\alpha f_{2j}(\phi)\,\mathrm{d}\phi = 0 \tag{2.11}$$

for every i. Consider the rational function

$$G(z) = 1 + \sum_j \frac{z\lambda_{2j}}{1 - z\lambda_{2j}} f_{2j}(\alpha) \int_{-\alpha}^\alpha f_{2j}(\phi)\,\mathrm{d}\phi. \tag{2.12}$$

This function has zeros at $z = 1/\lambda_{2i+1}$ from equation (2.11), has poles at $z = 1/\lambda_{2j}$ and is 1 at $z = 0$. Therefore,

$$G(z) = \prod_j \frac{(1 - z\lambda_{2j+1})}{(1 - z\lambda_{2j})} \tag{2.13}$$

i.e.

$$\frac{F_-(z,\alpha)}{F_+(z,\alpha)} = 1 + \sum_j \frac{z\lambda_{2j}}{1 - z\lambda_{2j}} f_{2j}(\alpha) \int_{-\alpha}^\alpha f_{2j}(\phi)\,\mathrm{d}\phi. \tag{2.14}$$

This is equation (A.16.6) of [1].

Starting with equation (2.2) one can similarly determine the coefficients d_{ij},

$$d_{ij} = \frac{-2\lambda_{2j+1}}{\lambda_{2i} - \lambda_{2j+1}} f_{2i}(\alpha) f_{2j+1}(\alpha) \tag{2.15}$$

and similarly,

$$f_{2i}(\alpha)\left[1 + \sum_j \frac{2\lambda_{2j+1}}{\lambda_{2i} - \lambda_{2j+1}} f_{2j+1}(\alpha) \int_0^\alpha f_{2j+1}(\phi)\,\mathrm{d}\phi\right] = f_{2i}(0). \tag{2.16}$$

However, from equations (1.13) and (2.2),

$$\begin{aligned} \lambda_{2i} f_{2i}(0) &= \int_{-\alpha}^\alpha S_+(0,\phi) f_{2i}(\phi)\,\mathrm{d}\phi = \int_{-\alpha}^\alpha S(\theta) f_{2i}(\theta)\,\mathrm{d}\theta \\ &= \int_{-\alpha}^\alpha S(\theta)\left[f_{2i}(0) + \sum_j d_{ij} \int_0^\theta f_{2j+1}(\phi)\,\mathrm{d}\phi\right]\mathrm{d}\theta \end{aligned} \tag{2.17}$$

or, substituting for d_{ij} from equation (2.15),

$$(\lambda_{2i} - 2I(\alpha))\, f_{2i}(0) = \sum_j \frac{-2\lambda_{2j+1}}{\lambda_{2i} - \lambda_{2j+1}} f_{2i}(\alpha) f_{2j+1}(\alpha) \int_{-\alpha}^\alpha \mathrm{d}\theta\, S(\theta) \int_0^\theta \mathrm{d}\phi\, f_{2j+1}(\phi) \tag{2.18}$$

where

$$2I(\theta) = \int_{-\theta}^{\theta} S(\phi)\, \mathrm{d}\phi = \int_{-\theta}^{\theta} S_+(\phi)\, \mathrm{d}\phi = 2\int_0^{\theta} S(\phi)\, \mathrm{d}\phi. \tag{2.19}$$

Now a partial integration gives

$$\begin{aligned}\int_{-\alpha}^{\alpha} \mathrm{d}\theta\, S(\theta) \int_0^{\theta} \mathrm{d}\phi\, f_{2j+1}(\phi) &= 2\int_0^{\alpha} \mathrm{d}\theta\, S(\theta) \int_0^{\alpha} \mathrm{d}\phi\, f_{2j+1}(\phi) - 2\int_0^{\alpha} \mathrm{d}\theta\, f_{2j+1}(\theta) \int_0^{\theta} \mathrm{d}\phi\, S(\phi)\\ &= \int_0^{\alpha} \mathrm{d}\theta\, f_{2j+1}(\theta)[2I(\alpha) - 2I(\theta)]\end{aligned} \tag{2.20}$$

so that from equations (2.16) and (2.18), removing the common factor $f_{2i}(\alpha)$ and rearranging,

$$\lambda_{2i} - 2I(\alpha) + \sum_j \frac{2\lambda_{2j+1}}{\lambda_{2i} - \lambda_{2j+1}} f_{2j+1}(\alpha) \int_0^{\alpha} \mathrm{d}\theta\, f_{2j+1}(\theta)\, [\lambda_{2i} - 2I(\theta)]\, \mathrm{d}\theta = 0. \tag{2.21}$$

In other words, the function

$$1 - 2zI(\alpha) + \sum_j \frac{2z\lambda_{2j+1}}{1 - z\lambda_{2j+1}} f_{2j+1}(\alpha) \int_0^{\alpha} \mathrm{d}\theta\, f_{2j+1}(\theta)[1 - 2zI(\theta)] \tag{2.22}$$

has zeros at $z = 1/\lambda_{2i}$, has poles at $1/\lambda_{2j+1}$ and is 1 at $z = 0$, so that it is equal to $F_+(z,\alpha)/F_-(z,\alpha)$, i.e.

$$\frac{F_+(z,\alpha)}{F_-(z,\alpha)} = 1 - 2zI(\alpha) + \sum_j \frac{2z\lambda_{2j+1}}{1 - z\lambda_{2j+1}} f_{2j+1}(\alpha) \int_0^{\alpha} \mathrm{d}\theta\, f_{2j+1}(\theta)[1 - 2zI(\theta)]. \tag{2.23}$$

Equation (2.14) is simpler than (2.23) since instead of (2.19) one has

$$\int_{-\theta}^{\theta} S_-(\phi)\, \mathrm{d}\phi = 0. \tag{2.24}$$

3. Relation between $F_1(z,t)$, $F_+(z,t)$ and $F_-(z,t)$

As for $\beta = 2$ the even and odd solutions of equation (1.3) are also solutions of equation (1.13), similarly for $\beta = 1$, the solutions of equation (1.3) are also solutions of

$$\mu \begin{bmatrix} \xi(\theta) \\ \eta(\theta) \end{bmatrix} = \int_{-\alpha}^{\alpha} \begin{bmatrix} S_\pm(\theta,\phi) & D_\mp(\theta,\phi) \\ J_\pm(\theta,\phi) & S_\mp(\theta,\phi) \end{bmatrix} \begin{bmatrix} \xi(\phi) \\ \eta(\phi) \end{bmatrix} \mathrm{d}\phi \tag{3.1}$$

with

$$S_\pm(\theta,\phi) = \tfrac{1}{2}[S(\theta,\phi) \pm S(\theta,-\phi)] \tag{3.2}$$

$$D_\pm(\theta,\phi) = \tfrac{1}{2}[D(\theta,\phi) \pm D(\theta,-\phi)] \tag{3.3}$$

$$J_\pm(\theta,\phi) = \tfrac{1}{2}[J(\theta,\phi) \pm J(\theta,-\phi)]. \tag{3.4}$$

Similar to equation (2.3), we have

$$\frac{\partial}{\partial\theta} D_\pm(\theta,\phi) = -\frac{\partial}{\partial\phi} D_\mp(\theta,\phi) \tag{3.5}$$

$$\frac{\partial}{\partial\theta} J_\pm(\theta,\phi) = -\frac{\partial}{\partial\phi} J_\mp(\theta,\phi). \tag{3.6}$$

The kernel $S_+(\theta,\phi)$ ($S_-(\theta,\phi)$) acting on any function $g(\phi)$ selects the even (odd) part of g and the result is an even (odd) function:

$$\int_{-\alpha}^{\alpha} S_\pm(\theta,\phi)g(\phi)\,\mathrm{d}\phi = \int_{-\alpha}^{\alpha} S_\pm(\theta,\phi)\tfrac{1}{2}[g(\phi)\pm g(-\phi)]\,\mathrm{d}\phi \tag{3.7}$$

is an even (odd) function of θ. Therefore, in the operator sense

$$S_+\circ S_- = S_-\circ S_+ = 0. \tag{3.8}$$

Similarly, $D_+(\theta,\phi)$ and $J_+(\theta,\phi)$ ($D_-(\theta,\phi)$ and $J_-(\theta,\phi)$) acting on any function $g(\phi)$ selects the even (odd) part of g and the result is an odd (even) function, so that

$$\begin{aligned}&D_+\circ J_+ = D_-\circ J_- = D_-\circ S_+ = D_+\circ S_- = J_+\circ S_- = J_-\circ S_+ = 0\\ &J_+\circ D_+ = J_-\circ D_- = S_+\circ D_+ = S_-\circ D_- = S_+\circ J_+ = S_-\circ J_- = 0\end{aligned} \tag{3.9}$$

i.e.

$$\sigma_+\circ\sigma_- = \sigma_-\circ\sigma_+ = 0 \tag{3.10}$$

where

$$\sigma_\pm(\theta,\phi) = \begin{bmatrix} S_\pm(\theta,\phi) & D_\mp(\theta,\phi)\\ J_\pm(\theta,\phi) & S_\mp(\theta,\phi)\end{bmatrix}. \tag{3.11}$$

The kernels σ_+ and σ_- have the same set of eigenvalues, half of them being zero. In what follows we will be concerned with only the non-zero eigenvalues.

Equation (3.1) written in full reads for the upper sign, for example,

$$\mu\xi(\theta) = \int_{-\alpha}^{\alpha}[S_+(\theta,\phi)\xi(\phi) + D_-(\theta,\phi)\eta(\phi)]\,\mathrm{d}\phi \tag{3.12}$$

$$\mu\eta(\theta) = \int_{-\alpha}^{\alpha}[J_+(\theta,\phi)\xi(\phi) + S_-(\theta,\phi)\eta(\phi)]\,\mathrm{d}\phi. \tag{3.13}$$

Differentiating equation (3.13) with respect to θ, and comparing with (3.12), one gets

$$xi(\theta) = \frac{\mu}{\mu-1}\eta'(\theta). \tag{3.14}$$

Thus, $\xi(\theta)$ and $\eta(\theta)$ have opposite parities and $\xi(\theta)$ is proportional to the derivative of $\eta(\theta)$. With the upper (lower) sign in (3.1), $\xi(\theta)$ is even (odd) and $\eta(\theta)$ is odd (even).

Now from equation (2.3) and a partial integration,

$$\begin{aligned}\int_{-\alpha}^{\alpha} D_-(\theta,\phi)\eta(\phi)\,\mathrm{d}\phi &= \frac{\partial}{\partial\theta}\int_{-\alpha}^{\alpha} S_-(\theta,\phi)\eta(\phi)\,\mathrm{d}\phi = -\int_{-\alpha}^{\alpha}\frac{\partial}{\partial\phi}S_+(\theta,\phi)\eta(\phi)\,\mathrm{d}\phi\\ &= -2S_+(\theta,\alpha)\eta(\alpha) + \int_{-\alpha}^{\alpha} S_+(\theta,\phi)\eta'(\phi)\,\mathrm{d}\phi.\end{aligned} \tag{3.15}$$

For $\eta(\theta)$ an odd function, this gives with equations (3.12) and (3.14)

$$\frac{\mu^2}{\mu-1}\eta'(\theta) = -2S_+(\theta,\alpha)\eta(\alpha) + \left(\frac{\mu}{\mu-1}+1\right)\int_{-\alpha}^{\alpha} S_+(\theta,\phi)\eta'(\phi)\,\mathrm{d}\phi. \tag{3.16}$$

Substituting the expansion of $\eta'(\theta)$ in terms of the $f_{2j}(\theta)$,

$$\eta'(\theta) = \sum_i c_i f_{2i}(\theta) \qquad \eta(\theta) = \sum_i c_i\int_0^{\theta} f_{2i}(\phi)\,\mathrm{d}\phi \tag{3.17}$$

in equation (3.16), and using (2.4)

$$\left(\frac{\mu^2}{\mu-1} - \frac{2\mu-1}{\mu-1}\lambda_{2i}\right)c_i + 2\lambda_{2i}f_{2i}(\alpha)\sum_j c_j\int_0^{\alpha} f_{2j}(\phi)\,\mathrm{d}\phi = 0. \tag{3.18}$$

Therefore, the eigenvalues μ_i are the roots of the algebraic equation

$$\det\left[\left(\mu^2-(2\mu-1)\lambda_{2i}\right)\delta_{ij}+2(\mu-1)\lambda_{2i}f_{2i}(\alpha)\int_0^\alpha f_{2j}(\phi)\,\mathrm{d}\phi\right]=0. \quad (3.19)$$

The function $F_1(z,\alpha)$ is, therefore, obtained by substituting $\mu=1/z$ in the left-hand side of this equation and multiplying by an appropriate power of z to remove all its negative powers,

$$\begin{aligned}F_1(z,\alpha)&=\det\left[\left(1-(2z-z^2)\lambda_{2i}\right)\delta_{ij}+2z(1-z)\lambda_{2i}f_{2i}(\alpha)\int_0^\alpha f_{2j}(\phi)\,\mathrm{d}\phi\right]\\&=\prod_i(1-(2z-z^2)\lambda_{2i})\left[1+\sum_j\frac{2z(1-z)\lambda_{2j}}{1-(2z-z^2)\lambda_{2j}}f_{2j}(\alpha)\int_0^\alpha f_{2j}(\phi)\,\mathrm{d}\phi\right].\end{aligned} \quad (3.20)$$

For the last equality note that

$$\begin{aligned}\det\left[a_i\delta_{ij}+b_ic_j\right]&=\det\begin{bmatrix}1 & c_j\\ 0 & a_i\delta_{ij}+b_ic_j\end{bmatrix}=\det\begin{bmatrix}1 & c_j\\ -b_i & a_i\delta_{ij}\end{bmatrix}\\&=\prod_i a_i\left(1+\sum_j\frac{b_jc_j}{a_j}\right).\end{aligned} \quad (3.21)$$

Now

$$\prod_i\left(1-(2z-z^2)\lambda_{2i}\right)=F_+(2z-z^2,\alpha) \quad (3.22)$$

while using relation (2.14)

$$\begin{aligned}1&+2z(1-z)\sum_i\frac{\lambda_{2i}}{1-(2z-z^2)\lambda_{2i}}f_{2i}(\alpha)\int_0^\alpha f_{2i}(\phi)\,\mathrm{d}\phi\\&=\frac{1}{2-z}\left(2-z+(1-z)\sum_i\frac{(2z-z^2)\lambda_{2i}}{1-(2z-z^2)\lambda_{2i}}f_{2i}(\alpha)\int_{-\alpha}^\alpha f_{2i}(\phi)\,\mathrm{d}\phi\right)\\&=\frac{1}{2-z}\left(1+(1-z)\frac{F_-(2z-z^2,\alpha)}{F_+(2z-z^2,\alpha)}\right)\end{aligned} \quad (3.23)$$

so that finally

$$(2-z)F_1(z,\alpha)=F_+(2z-z^2,\alpha)+(1-z)F_-(2z-z^2,\alpha). \quad (3.24)$$

This equation is equivalent to (1.20), since the left-hand side of equation (3.24) is

$$\begin{aligned}(2-z)F_1(z,\alpha)&=\sum_r\left((1-z)^rE_1(r,\alpha)+(1-z)^{r+1}E_1(r,\alpha)\right)\\&=\sum_r(1-z)^{2r}\left[E_1(2r,\alpha)+E_1(2r-1,\alpha)\right]\\&\quad+\sum_r(1-z)^{2r+1}\left[E_1(2r,\alpha)+E_1(2r+1,\alpha)\right]\end{aligned} \quad (3.25)$$

while on the right-hand side

$$\begin{aligned}F_+(2z-z^2,\alpha)&=\sum_r(1-2z+z^2)^rE_+(r,\alpha)\\&=\sum_r(1-z)^{2r}E_+(r,\alpha)\end{aligned} \quad (3.26)$$

$$(1-z)F_-(2z-z^2,\alpha) = (1-z)\sum_r (1-2z+z^2)^r E_-(r,\alpha)$$
$$= \sum_r (1-z)^{2r+1} E_-(r,\alpha). \tag{3.27}$$

Comparing the various powers of $(1-z)$, we get the equivalence of equations (1.20) and (3.24).

One could have started with the lower sign in equation (3.1), and used equation (2.23) to arrive at the same result.

4. Relation between $F_4(z,t)$ and $F_\pm(z,t)$

For the symplectic ensemble we can again separate $K(4;\theta,\phi)$ into even and odd parts,

$$K(4;\theta,\phi) = \sigma_+(\theta,\phi) + \sigma_-(\theta,\phi) \tag{4.1}$$

$$\sigma_\pm(\theta,\phi) = \begin{bmatrix} S_\pm(2\theta,2\phi) & D_\mp(2\theta,2\phi) \\ I_\pm(2\theta,2\phi) & S_\mp(2\theta,2\phi) \end{bmatrix} \tag{4.2}$$

where $S_\pm(\theta,\phi)$, $D_\pm(\theta,\phi)$ and $I_\pm(\theta,\phi)$ are given by equations (3.2), (3.3) and a similar equation

$$I_\pm(\theta,\phi) = \tfrac{1}{2}[I(\theta,\phi) \pm I(\theta,-\phi)]. \tag{4.3}$$

The eigenvalues of the integral equation (1.3) are again also the eigenvalues of an integral equation with the kernel either $\sigma_+(\theta,\phi)$ or $\sigma_-(\theta,\phi)$ and the components of the eigenfunctions have definite opposite parities. It is convenient to take 2θ and 2ϕ as new variables and write the integral equation (1.3) as

$$\mu \begin{bmatrix} \xi(\theta) \\ \eta(\theta) \end{bmatrix} = \frac{1}{2}\int_{-2\alpha}^{2\alpha} \begin{bmatrix} S_\pm(\theta,\phi) & D_\mp(\theta,\phi) \\ I_\pm(\theta,\phi) & S_\mp(\theta,\phi) \end{bmatrix} \begin{bmatrix} \xi(\phi) \\ \eta(\phi) \end{bmatrix} \mathrm{d}\phi. \tag{4.4}$$

Following section 3 we find now $\xi(\theta) = \eta'(\theta)$. The arguments proceed as in section 3; equations corresponding to (3.18) and (3.19) are now

$$(\mu - \lambda_{2i})c_i + \lambda_{2i} f_{2i}(2\alpha) \sum_j c_j \int_0^{2\alpha} f_{2j}(\phi)\,\mathrm{d}\phi = 0 \tag{4.5}$$

$$\det\left[(\mu-\lambda_{2i})\delta_{ij} + \lambda_{2i} f_{2i}(2\alpha)\int_0^{2\alpha} f_{2j}(\phi)\,\mathrm{d}\phi\right]$$
$$= \prod_i (\mu-\lambda_{2i})\left[1 + \sum_j \frac{\lambda_{2j}}{1-\lambda_{2j}} f_{2j}(2\alpha)\int_0^{2\alpha} f_{2j}(\phi)\,\mathrm{d}\phi\right] = 0 \tag{4.6}$$

so that with equation (2.14)

$$F_4(z,\alpha) = \prod_i (1-z\lambda_{2i})\left[1 + \sum_j \frac{z\lambda_{2j}}{1-z\lambda_{2j}} f_{2j}(2\alpha)\int_0^{2\alpha} f_{2j}(\phi)\,\mathrm{d}\phi\right]$$
$$= F_+(z,2\alpha)\left[\frac{1}{2} + \frac{1}{2}\frac{F_-(z,2\alpha)}{F_+(z,2\alpha)}\right]. \tag{4.7}$$

The final result is

$$F_4(z,\alpha) = \tfrac{1}{2}[F_+(z,2\alpha) + F_-(z,2\alpha)] \tag{4.8}$$

which is equation (1.21).

5. Conclusion

New proofs of the known equations (1.20) and (1.21) are given. They, along with equations (1.1) and (1.15)–(1.18), relate the spacing functions $E_\beta(r,t)$ for $\beta = 1$, 2 and 4. The known equation (2.14) and a new one (2.23) relating odd and even spheroidal functions are recovered. Equations (1.20) and (1.21) along with Painlevé equations have been useful to derive the asymptotic behaviour [4] of the spacing functions $E_\beta(r,t)$ among others.

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